JOURNAL OF MULTIVARIATE ANALYSIS 49, 97–109 (1994)

Bootstrapping Multivariate U-Quantiles and Related Statistics

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The asymptotic consistency of the bootstrap approximation of the vector of the marginal generalized quantiles of U-statistic structure (multivariate U-quantiles for short) is established. The asymptotic accuracy of the bootstrap approximation is also obtained. Extensions to smooth functions of marginal generalized quantiles are given and some specific examples, such as the vector of marginal sample quantiles and the vector of marginal Hodges–Lehmann location estimators, are discussed.

1. INTRODUCTION

In this paper we investigate Efron's bootstrap approximation for a wide class of multivariate generalized quantiles. We also look at the classical normal approximation for these multivariate statistics. Our results extend previous work by Choudhury and Serfling (1988) and Helmers, Janssen, and Veraverbeke (1992) for univariate U-quantiles to the multivariate case. Let $X_1 = (X_{11}, ..., X_{k1}), ..., X_N = (X_{1N}, ..., X_{kN})$ be independent k-dimensional random vectors defined on a single probability space $(\Omega, \mathcal{A}, P)$ having common distribution function (df) F on $\mathbb{R}^k$. Let $h_1(x_1, ..., x_m), ...,$

Received July 22, 1992; revised April 15, 1993.

AMS 1991 subject classifications: 62G09, 62G30, 62G20.

Keywords and phrases: quantiles, U-statistics, bootstrapping, asymptotic consistency, asymptotic accuracy, Hodges–Lehmann location estimator, estimator of spread, interquantile ranges.

* Research partially supported by a grant of The Netherlands Organization for Scientific Research (NWO).

0047-259X/94 $6.00

$h_q(x_1, ..., x_m)$ be kernels of degree m (i.e., real-valued functions symmetric in its m arguments) and let

$$H_F(y) = P(h_j(X_{i1}, ..., X_{im}) \leq y_{ij}, \qquad i = 1, ..., k, j = 1, ..., q), \tag{1.1}$$

with $y = (y_{ij}, i = 1, ..., k, j = 1, ..., q) \in \mathbb{R}^{kq}$, denote the joint df of the kq random variables

$$h_j(X_{i1}, ..., X_{im}), \qquad i = 1, ..., k, j = 1, ..., q. \tag{1.2}$$

In addition, let $H_{F,i,j}$ denote the marginal df of $h_j(X_{i1}, ..., X_{im})$ and, for $0 < p < 1$, let

$$\xi_{ij}(p) = H_{F,i,j}^{-1}(p) = \inf\{x : H_{F,i,j}(x) \geq p\} \tag{1.3}$$

denote the corresponding pth marginal quantile. Define the associated empirical df of U-statistic structure

$$H_{N,i,j}(y_{ij}) = \binom{N}{m}^{-1} \sum_{1 \leq v_1 < \cdots < v_m \leq N} \cdots \sum I(h_j(X_{iv_1}, ..., X_{iv_m}) \leq y_{ij}) \tag{1.4}$$

for real y_{ij} and $N \geq m$, and for $0 < p < 1$, let

$$\hat{\xi}_{ij}(p) = H_{N,i,j}^{-1}(p) \tag{1.5}$$

denote the corresponding pth empirical marginal quantile.

Statistics of the form (1.5) were considered by Helmers, Janssen, and Veraverbeke (1992) for the univariate case $k = q = 1$. Asymptotic consistency and the asymptotic accuracy of the bootstrap approximation for these univariate statistics (U-quantiles for short) was established. The present paper aims at extending these results to the multivariate case. Define *multivariate U-quantiles* by

$$(\hat{\xi}_{11}(p_{11}), ..., \hat{\xi}_{k1}(p_{k1}), ..., \hat{\xi}_{1q}(p_{1q}), ..., \hat{\xi}_{kq}(p_{kq})), \tag{1.6}$$

where $0 < p_{ij} < 1, i = 1, ..., k, j = 1, ..., q$. In the special case $q = m = 1$ and $h_1(x) = x$, (1.6) reduces to the k-vector of ordinary marginal quantiles studied by G. J. Babu and C. R. Rao (1988). These authors obtained asymptotic normality for this special case. In Section 2 we extend their result to the more general class of multivariate U-quantiles of the form (1.6). We also obtain a Berry–Esseen bound for these statistics. In Section 3 we derive the asymptotic consistency and establish the asymptotic accuracy of the bootstrap approximation for multivariate U-quantiles of type (1.6). Some useful extensions and specific examples are briefly discussed in Section 4.

2. NORMAL APPROXIMATION

Let Σ denote the symmetric $(kq \times kq)$-matrix with elements

$$\sigma_{ji,lr} = \operatorname{cov}\{P(h_j(X_{i1}, ..., X_{im}) \leqslant \xi_{ij}(p_{ij}) \mid X_{i1}),\quad P(h_l(X_{r1}, ..., X_{rm}) \leqslant \xi_{rl}(p_{rl}) \mid X_{r1})\} \tag{2.1}$$

with $i, r = 1, ..., k$; $j, l = 1, ..., q$. In addition, let

$$D_{H_F} = \operatorname{diag}(h_{F,1,1}(\xi_{11}(p_{11})), ..., h_{F,k,1}(\xi_{k1}(p_{k1})), ..., h_{F,1,q}(\xi_{1q}(p_{1q})), ..., h_{F,k,q}(\xi_{kq}(p_{kq}))), \tag{2.2}$$

where $h_{F,i,j}$ denotes the density of $H_{F,i,j}$. Let $\xi = (\xi_{ij}(p_{ij}), i = 1, ..., k, j = 1, ..., q)$ and $\hat{\xi} = (\hat{\xi}_{ij}(p_{ij}), i = 1, ..., k, j = 1, ..., q)$. Clearly ξ denotes our parameter vector of interest and $\hat{\xi}$ its empirical counterpart (cf. (1.6)).

THEOREM 2.1. (a) *Let $H_{F,i,j}$ be continuously differentiable with density $h_{F,i,j}(\xi_{ij}(p_{ij})) > 0$. Then, as $N \to \infty$,*

$$\sup_y |P(N^{1/2}(\hat{\xi} - \xi) \leqslant y) - \Phi(y; 0, m^2 D_{H_F}^{-1/2} \Sigma D_{H_F}^{-1/2})| \to 0, \tag{2.3}$$

where $\Phi(\cdot; 0, V)$ is the qk-variate normal distribution with mean vector 0 *and covariance matrix V and $y = (y_{11}, ..., y_{k1}, ..., y_{1q}, ..., y_{kq})'$.*

(b) *In addition, suppose that $h_{F,i,j}$ satisfies a Lipschitz condition of order* 1 *on a neighborhood of $\xi_{ij}(p_{ij})$, $i = 1, ..., k$, $j = 1, ..., q$, and Σ is positive definite. Then, as $N \to \infty$,*

$$\sup_y |P(N^{1/2}(\hat{\xi} - \xi) \leqslant y) - \Phi(y; 0, m^2 D_{H_F}^{-1/2} \Sigma D_{H_F}^{-1/2})| = O(N^{-1/2}). \tag{2.4}$$

Proof. To begin with we remark that

$$\begin{aligned}
P(N^{1/2}(\hat{\xi} - \xi) \leqslant y) &= P(N^{1/2}(\hat{\xi}_{ij}(p_{ij}) - \xi_{ij}(p_{ij})) \leqslant y_{ij}, i = 1, ..., k, i = 1, ..., q) \\
&= P(H_{N,i,j}^{-1}(p_{ij}) \leqslant H_{F,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2}, i = 1, ..., k, j = 1, ..., q) \\
&= P(H_{N,i,j}(H_{F,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2}) \geqslant p_{ij}, i = 1, ..., k, j = 1, ..., q) \\
&= P(N^{1/2}\{H_{N,i,j}(H_{F,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2}) \\
&\quad - H_{F,i,j}(H_{F,i,j}^{-1}(p_{ij}) + N^{-1/2} y_{ij})\} \\
&\quad \geqslant N^{1/2}\{p_{ij} - H_{F,i,j}(H_{F,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2})\}, i = 1, ..., k, j = 1, ..., q).
\end{aligned} \tag{2.5}$$

The smoothness assumption of $H_{F,i,j}$ ensures that

$$\sqrt{N}\,\{p_{ij}-H_{F,i,j}(H_{F,i,j}^{-1}(p_{ij})+N^{-1/2}y_{ij})\}\to -y_{ij}h_{F,i,j}(\xi_{ij}(p_{ij})) \tag{2.6}$$

a.s. [P], for $i=1, ..., k$, $i=1, ..., q$. Next note that the random variable (r.v.) $T_{N,i,j}(y_{ij})$ defines as

$$\sqrt{N}\,\{H_{N,i,j}(H_{F,i,j}^{-1}(p_{ij})+y_{ij}N^{-1/2})-H_{F,i,j}(H_{F,i,j}^{-1}(p_{ij})+y_{ij}N^{-1/2})\} \tag{2.7}$$

is a normalized U-statistic of degree m with bounded kernel, depending on n, of the form

$$\begin{aligned} h_{N,i,j}&(x_1, ..., x_m; y_{ij})\\ &=I\{h_j(x_1, ..., x_m)\leqslant \xi_{ij}(p_{ij})+y_{ij}N^{-1/2}\}\\ &\quad -P(h_j(X_{i1}, ..., X_{im})\leqslant \xi_{ij}(p_{ij})+y_{ij}N^{-1/2}) \end{aligned} \tag{2.8}$$

with $(x_1, ..., x_m)\in\mathbb{R}^m$, for $i=1, ..., k$, $j=1, ..., q$. At this point we invoke an easy modification of the theorem on page 188 of Serfling (1980) to find that for $i=1, ..., k$, $j=1, ..., q$

$$E\left\{T_{N,i,j}(y_{ij})-\sum_{s=1}^{N} E(T_{N,i,j}(y_{ij})\mid X_{is})\right\}^2=O(N^{-1}). \tag{2.9}$$

In view of (2.5)–(2.9) and Chebychev's inequality it now clearly suffices to check that the kq-vector given by

$$\sum_{s=1}^{N} E(T_{N,i,j}(y_{ij})\mid X_{is}), \qquad i=1, ..., k, j=1, ..., q \tag{2.10}$$

converges in distribution to $\Phi(\cdot; 0, \Sigma)$, as $n\to\infty$, with Σ as in (2.1). But this is an easy matter, because the random vector (2.10) is a normalized sum of i.i.d. random summands, with zero mean and covariance matrix V_N. Clearly

$$\begin{aligned} E(T_{N,i,j}&(y_{ij})\mid X_{is})\\ &=\frac{m}{\sqrt{N}}\sum_{s=1}^{N}\{P(h_j(X_{is}, X_{iv_2}, ..., X_{iv_m})\leqslant \xi_{ij}(p_{ij})+y_{ij}N^{-1/2}\mid X_{is})\\ &\quad -P(h_j(X_{i1}, ..., X_{im})\leqslant \xi_{ij}(p_{ij})+y_{ij}N^{-1/2})\}, \qquad i=1, ..., k, j=1, ..., q, \end{aligned} \tag{2.11}$$

where $1\leqslant v_2<\cdots<v_m\leqslant N$ can be chosen arbitrary, provided $v_l\neq s$, $l=2, ..., m$. The $(kq\times kq)$-matrix V_N is precisely the covariance matrix of

the kq-vector (2.11). It remains to show that V_N approaches Σ, as $N \to \infty$. The simple relation

$$\sigma^2\left\{\sum_{s=1}^{N} E((T_{N,i,j}(Y_{ij}) - T_{N,i,j}(0)) \mid X_{is})\right\}$$
$$\leqslant m^2\sigma^2\{P(h_j(X_{i1}, ..., X_{im}) \in (\xi_{ij}(p_{ij}), \xi_{ij}(p_{ij}) + y_{ij}N^{-1/2}) \mid X_{i1})\}$$
$$= O(N^{-1}), \quad \text{as} \quad N \to \infty, \tag{2.12}$$

can be used. Here we have employed the mean value theorem (applied to $H_{F,i,j}$ on a neighborhood of $\xi_{ij}(p_{ij})$) and a well-known property of conditional moments. Since the kq-vector given by

$$\sum_{s=1}^{N} E(T_{N,i,j}(0) \mid X_{is}), \qquad i = 1, ..., k, j = 1, ..., q \tag{2.13}$$

has covariance matrix Σ (cf. (2.1)) our proof of relation (2.3) is now completed.

It remains to establish the Berry–Esseen bound (2.4). To do this we apply Theorem 1.16 of Götze (1987) to the kq-vector (cf. (2.7)) of U-statistic type

$$T_{N,i,j}(y_{ij}), \qquad i = 1, ..., k, j = 1, ..., q. \tag{2.14}$$

The conditions of Götze's theorem are easily verified, since by assumption Σ is positive definite and the kernel functions $h_{N,i,j}$ (cf. (2.8)) are all bounded by 1. In addition, relation (2.6) is now replaced by the stronger assertion that

$$\sqrt{N}\,\{p_{ij} - H_{F,i,j}(H_{F,i,j}^{-1}(p_{ij}) + y_{ij}N^{-1/2})\} + y_{ij}h_{F,i,j}(\xi_{ij}(p_{ij})) = O(N^{-1/2}) \tag{2.15}$$

as $N \to \infty$. For this we use the Lipschitz condition on $h_{F,i,j}$. This completes the proof of (2.4) and the theorem is proved. ▮

Relation (2.3) extends related results of Choudhury and Serfling (1988) and Babu and Rao (1988) for the univariate case $k = q = 1$, respectively the case of ordinary marginal quantiles ($q = m = 1$, $h_1(x) = x$), to a considerably more general class of statistics. Relation (2.4) supplements all this with a Berry–Esseen bound. We note in passing that the strong consistency of multivariate U-quantiles $\hat{\xi}$ in estimating the parameter of interest ξ follows directly from Corollary 3.2 of Helmers, Janssen, and Serfling (1988), provided ξ is uniquely determined.

In applications one often wishes to establish a confidence region for ξ and a studentized version of (2.3) is required; i.e., a consistent estimator of the covariance matrix in (2.3) is needed. An alternative approach is to employ bootstrap methods for the construction of a confidence region for ξ. In Section 3 we establish a bootstrap analog of (2.3) under a slightly more stringent smoothness condition on H_F. With the aid of this result various bootstrap based confidence regions for ξ can easily be constructed.

3. Bootstrap Approximation

Let $\hat{F}_N$ denote the k-variate empirical *df* based on $X_1, ..., X_N$. Define

$$H^*_{N,i,j}(y_{ij}) = \binom{N}{m}^{-1} \sum_{1 \leqslant v_1 < \cdots < v_k \leqslant N} \cdots \sum I(h_j(X^*_{iv_1}, ..., X^*_{iv_m}) \leqslant y_{ij}) \qquad (3.1)$$

with $y_{ij} \in \mathbb{R}$, the marginal empirical *df* of U-statistic structure based on the (marginal) bootstrap sample $X^*_{i1}, ..., X^*_{iN}$, i.e., the ith component of $X^*_1, ..., X^*_N$. Here and elsewhere $X^*_1, ..., X^*_N$ denotes a random sample drawn with replacement from $\hat{F}_N$, conditionally given $X_1, ..., X_N$. Define $\hat{\xi}^*_{ij}(p) = H^{*-1}_{N,i,j}(p)$, $0 < p < 1$, with $H^*_{N,i,j}$ as in (3.1). Let P^*_N denote probability corresponding to $\hat{F}_N$.

Theorem 3.1. (a) *Let $H_{F,i,j}$ be continuously differentiable (with density $h_{F,i,j}$) on a neighborhood of $\xi_{p_{ij}}$ with $h_{F,i,j}(\xi_{ij}(p_{ij})) > 0$. Then, for almost every sample sequence $X_1, X_2, ...$,*

$$\sup_y |P^*_N(\sqrt{N}\,(\hat{\xi}^* - \xi) \leqslant y) - \Phi(y; 0, m^2 D^{-1/2}_{H_F} \Sigma D^{1/2}_{H_F})|$$
$$\to 0, \qquad \text{as} \quad N \to \infty. \qquad (3.2)$$

(b) *In addition, if $h_{F,i,j}$ satisfies a Lipschitz condition of order 1 on a neighborhood of $\xi_{ij}(p_{ij})$, $i = 1, ..., k, j = 1, ..., q$, and Σ is positive definite, then*

$$\sup_y |P^*_N(\sqrt{N}\,(\hat{\xi}^* - \xi) \leqslant y) - P(\sqrt{N}\,(\hat{\xi} - \xi) \leqslant y)| = O(N^{-1/4}(\ln N)^{3/4}) \qquad (3.3)$$

a.s. [P], *as $N \to \infty$.*

Proof. Similarly as in the proof of Theorem 2.1 of Helmers, Janssen, and Veraverbeke (1992) we write

$$P^*_N(\sqrt{N}\,(\hat{\xi}^* - \hat{\xi}) \leqslant y) = P^*_N(W^*_{N,i,j}(y_{ij}) \geqslant -D_{N,i,j}, i = 1, ..., k, j = 1, ..., q), \qquad (3.4)$$

where

$$W^*_{N,i,j}(y_{ij}) = \sqrt{N}\,\{H^*_{N,i,j}(H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}) - \bar{H}_{N,i,j}(H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}))\} \tag{3.5}$$

and

$$D_{N,i,j} = \sqrt{N}\,\{\bar{H}_{N,i,j}(H^{-1}_{N,i,j}(p_{ij})) + y_{ij}N^{-1/2}) - p_{ij}\} \tag{3.6}$$

with

$$\bar{H}_{N,i,j}(y_{ij}) = N^{-m} \sum_{v_1=1}^{N} \cdots \sum_{v_m=1}^{N} I(h_j(X_{iv_1}, ..., X_{iv_m}) \leqslant y_{ij}) \tag{3.7}$$

for $i = 1, ..., k, j = 1, ..., q$. As in Helmers, Janssen, and Veraverbeke(1992) we obtain

$$D_{N,i,j} \to y_{ij}h_{F,i,j}(\xi_{ij}(p_{ij})) \qquad \text{a.s. [P]} \tag{3.8}$$

as $N \to \infty$, $i = 1, ..., k, j = 1, ..., q$.

To proceed we deal with the random variables $W^*_{N,i,j}(y_{ij})$ defined in (3.5). Note that

$$\begin{aligned} W^*_{N,i,j}(y_{ij}) = \sqrt{N}\binom{N}{m}^{-1} &\sum_{v_1<\cdots<v_m}\cdots\sum I\{h_j(X^*_{iv_1}, ..., X^*_{iv_m}) \\ &\leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}\} - N^{-m}\sum_{s_1}\cdots\sum_{s_m} I\{h_j(X_{is_1}, ..., X_{is_m}) \\ &\leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}\}), \qquad i = 1, ..., k, j = 1, ..., q \end{aligned} \tag{3.9}$$

is of U-statistic structure, with kernel

$$\begin{aligned} g^*_{N,i,j}(x_1, ..., x_m; y_{ij}) = I\{h_j(x_1, ..., x_m) &\leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}\} \\ &- \frac{1}{N^m}\sum_{s_1}\cdots\sum_{s_m} I\{h_j(X_{is_1}, ..., X_{is_m}) \\ &\leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}\} \end{aligned} \tag{3.10}$$

for all $(x_1, ..., x_m) \in \mathbb{R}^m$. The next step is to check that an order bound similar to (2.11) holds true here too. Because the $g^*_{N,i,j}$'s are bounded by 1 in absolute value we indeed have that, for each $r \geqslant 2$,

$$E^*_N(T^*_{N,i,j}(y_{ij}) - \sum_{s=1}^{N} E^*_N(T^*_{N,i,j}(y_{ij}) \mid X^*_{is}))^{2r} = O(N^{-r}) \tag{3.11}$$

a.s. [P], as $N \to \infty$. Here $T^*_{N,i,j}(y_{ij})$ is defined by (2.7) with X_{ij} replaced by X^*_{ij}, $H_{F,i,j}$ by $H_{N,i,j}$, and $H_{N,i,j}$ by $H^*_{N,i,j}$.

Similarly as in the proof of Theorem 2.1, relation (2.3), our problem reduces to one of proving the result

$$\begin{aligned} P^*_N\Bigg(\sum_{s=1}^{N} \frac{m}{\sqrt{N}} & (P^*_N(h_j(X^*_{is}, X^*_{iv_1}, ..., X^*_{iv_{m-1}}) \\ & \leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2} \mid X^*_{is}) - P^*_N(h_j(X^*_{i1}, ..., X^*_{im}) \\ & \leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}) \leqslant z_{ij},\ i=1, ..., k, j=1, ..., q) \\ & - \Phi(z; 0, N \operatorname{var}^*_N\{E^*_N(T^*_N(y) \mid X^*_1))\}\Bigg) \to 0 \end{aligned} \tag{3.12}$$

a.s. [P], as $N \to \infty$, where we have used the fact that

$$\begin{aligned} \sum_{s=1}^{N} & E^*_N(T^*_{N,i,j}(y_{ij}) \mid X^*_{is}) \\ &= \sum_{s=1}^{N} \left(\sqrt{N}\binom{N}{m}\right)^{-1} \sum_{v_1 < \cdots < v_m} \cdots \sum \\ &\quad E^*_N\Bigg(\bigg(I\{h_j(X^*_{iv_1}, ..., X^*_{iv_m}) \leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}\} \\ &\quad - N^{-m} \sum_{s_1} \cdots \sum_{s_m} I\{h_j(X_{is_1}, ..., X_{is_m}) \leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}\}\bigg) \bigg| X^*_{is}\Bigg) \\ &= \frac{m}{\sqrt{N}} \sum_{s=1}^{N} (P^*_N(h_j(X^*_{is}, X^*_{iv_2}, ..., X^*_{iv_m}) \leqslant H^{-1}_{N,j}(p_{ij}) + y_{ij}N^{-1/2} \mid X^*_{is}) \\ &\quad - P^*_N(h_j(X^*_{i1}, ..., X^*_{im}) \leqslant H_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2} \mid X^*_{is})), \end{aligned} \tag{3.13}$$

where $s \notin \{v_2, ..., v_m\}$ and $v_2 < \cdots < v_m$ in the next to last line. To proceed we note that (3.13) in turn is equal to

$$\begin{aligned} \frac{m}{\sqrt{N}} \sum_{s=1}^{N} N^{-m+1} \sum_{s_1} \cdots \sum_{s_{m-1}} & \Bigg(I\{h_j(X^*_{is}, X_{is_1}, ..., X_{is_{m-1}}) \\ & \leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}\} - \frac{1}{N} \sum_{l=1}^{N} I\{(h_j(X_{il}, X_{is_1}, ..., X_{is_{m-1}}) \\ & \leqslant H^{-1}_{N,i,j}(p_{ij}) + y_{ij}N^{-1/2}\}\Bigg). \end{aligned} \tag{3.14}$$

Also, note that the expression withing curly brackets appearing in the Var_N^* matrix in (3.12) is nothing but the kq-vector.

$$E_N^*(T_N^*(y) \mid X_1^*) = (E_N^*(T_{N,i,j}^*(y_{ij}) \mid X_{i1}^*),\ i = 1, ..., k, j = 1, ..., q).$$

Clearly (3.12) is true, since $E^*(T_N^*(y) \mid X_s^*)$ is a sum of i.i.d. random vectors (conditionally given $X_1, ..., X_N$). It remains, however, to investigate the covariance matrix of $E_N^*(T_N^*(y) \mid X_1^*)$, conditionally given $X_1, ..., X_N$. To do this, we first check that

$$\begin{aligned} &N \operatorname{var}_N^* \{E_N^*(T_{N,i,j}^*(y_{ij}) \mid X_s^*) \\ &\quad - E_N^*(T_{N,i,j}^*((H_{F,i,j}^{-1}(p_{ij}) - H_{N,i,j}^{-1}(p_{ij}))\sqrt{N}) \mid X_s^*)\} \end{aligned} \tag{3.15}$$

is sufficiently small. In fact, we have

$$\begin{aligned} &N \operatorname{var}_N^* \{E_N^*(T_{N,i,j}^*(y_{ij}) - T_{N,i,j}^*(\sqrt{N}\,(H_{F,i,j}^{-1}(p_{ij}) \\ &\quad - H_{N,i,j}^{-1}(p_{ij}))) \mid X_{is}^*\} \leqslant m^2 E_N^*(P_N^*(h_j(X_{i1}^*, ..., X_{im}^*) \\ &\quad \in (H_{N,i,j}^{-1}(p_{ij}), H_{N,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2}) \mid X_{is}^*))^2 \\ &= m^2 \cdot N^{-1} \sum_{\nu_1} (N^{-m+1} \sum_{\nu_2} \cdots \sum_{\nu_m} I\{h_j(X_{i\nu_1}, ..., X_{i\nu_m}) \\ &\quad \in (H_{N,i,j}^{-1}(p_{ij}), H_{N,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2})\})^2. \end{aligned} \tag{3.16}$$

At this point we use the almost sure bound (cf. Corollary 2.1 of Helmers, Janssen, and Serfling, 1988, p. 78)

$$\limsup_{N \to \infty} \left(\frac{N}{\log N}\right)^{1/2} \sup_x |H_{F,i,j}(x) - H_{N,i,j}(x)| \leqslant C_m \tag{3.17}$$

a.s. [P], as $N \to \infty$, with C_m as in the corollary, as well as the fact that

$$\begin{aligned} &P_N^*(h_j(X_{i1}^*, ..., X_{im}^*) \in (H_{N,i,j}^{-1}(p_{ij}), H_{N,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2}) \mid X_{i1}^*) \\ &\quad = m \frac{m!}{N^{m-1}} \sum_{\nu_2 < \cdots < \nu_m} I\{h_j(X_{i\nu_2}^*, ..., X_{i\nu_m}^*) \in (H_{N,i,j}^{-1}(p_{ij}), H_{N,i,j}^{-1}(p_{ij}) \\ &\quad + y_{ij} N^{-1/2})\} + O(N^{-1}) \end{aligned} \tag{3.18}$$

a.s. [P], as $N \to \infty$. Also, we have

$$\begin{aligned} &E\Bigg[N^{-m+1} \sum_{\nu_2 < \cdots < \nu_m} \cdots \sum (I\{h_j(x, X_{i\nu_2}, ..., X_{i\nu_m}) \\ &\quad \in (H_{N,i,j}^{-1}(p_{ij}), H_{N,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2})\} - P(h_j(x, X_{i\nu_2}, ..., X_{i\nu_m}) \\ &\quad \in (H_{N,i,j}^{-1}(p_{ij}), H_{N,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2}))\Bigg]^{2r} = O(N^{-r}) \end{aligned} \tag{3.19}$$

for any integer $r \geqslant 2$ (cf. Serfling, 1980, p. 185), and the easily verified fact that

$$P(h_j(x, X_{i\nu_2}, ..., X_{i\nu_m}) \in (H_{F,i,j}^{-1}(p_{ij}), H_{N,i,j}^{-1}(p_{ij}) + y_{ij}N^{-1/2})) \to 0, \quad \text{as} \quad N \to \infty. \quad (3.20)$$

We can conclude that

$$N \operatorname{var}_N^*\{E_N^*(T_{N,i,j}^*(y_{ij}) - T_{N,i,j}^*((H_{F,i,j}^{-1}(p_{ij}) - H_{N,i,j}(p_{ij}))\sqrt{N}) \mid X_{is}^*\} \to 0 \quad (3.21)$$

a.s. [P], as $N \to \infty$. In other words: the quantity (3.15) is indeed small enough for our purposes.

To complete our proof of (3.2) we must still show that

$$\frac{m^2}{N} \sum_{\nu_1} N^{-m+1} \sum_{\nu_2} \cdots \sum_{\nu_m} I\{h_j(X_{i\nu_1}, ..., X_{i\nu_m}) \leqslant H_{F,i,j}^{-1}(p_{ij})\}^2$$
$$\to m^2 E(P(h_j(X_{i1}, ..., X_{im}) \leqslant H_{F,i,j}^{-1}(p_{ij}) \mid X_{i1}))^2 \quad (3.22)$$

a.s. [P], as $N \to \infty$. But this follows from the fact that

$$N^{-m+1} \sum_{\nu_2} \cdots \sum_{\nu_m} I\{h_j(X_{i\nu_1}, ..., X_{i\nu_m}) \leqslant H_{F,i,j}^{-1}(p_{ij})\}$$
$$= \frac{(m-1)!}{N^{m-1}} \sum_{\nu_2} \cdots \sum_{\nu_m} I\{h_j(X_{i\nu_2}, ..., X_{i\nu_m}) \leqslant H_{F,i,j}^{-1}(p_{ij})\}. \quad (3.23)$$

A similar argument gives the desired convergences for the other elements of the covariance matrix in (3.12). This completes the proof of (3.2).

Next we establish the a.s. rate of convergence asserted in (3.3). Our argument follows the pattern of proof given in Theorem 3.1 of Helmers, Janssen, and Veraverbeke (1992). Quite similarly, we write

$$\sup_y |P_N^*(N^{1/2}(\hat{\xi}^* - \xi) \leqslant y) - P(N^{1/2}(\hat{\xi} - \xi) \leqslant y)| \leqslant \sum_{i=1}^{3} I_{iN}, \quad (3.24)$$

where, for some suitable constant $K > 0$,

$$I_{1N} = \sup_{\|y\| \leqslant K(\log N)^{1/2}} \left| P_N^*(N^{1/2}(\hat{\xi}^* - \hat{\xi}) \leqslant y) - \Phi\left(y; 0, m^2 D_{H_F}^{-1/2} \sum D_{H_F}^{-1/2}\right)\right| \quad (3.25)$$

$$I_{2N} = \sup_{\|y\| > K(\log N)^{1/2}} \left| P_N^*(N^{1/2}(\hat{\xi}^* - \hat{\xi}) \leqslant y) - \Phi\left(y; 0, m^2 D_{H_F}^{-1/2} \sum D_{H_F}^{-1/2}\right)\right| \quad (3.26)$$

and

$$I_{3N} = \sup_{y} \left| P(N^{1/2}(\hat{\xi} - \xi) \leqslant y) - \Phi\left(y; 0, m^2 D_{H_F}^{-1/2} \sum D_{H_F}^{-1/2} \right) \right| \quad (3.27)$$

where $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^{kq}$. Because of Theorem 2.1, relation (2.4), we know that $I_{3N} = O(N^{-1/2})$. To treat I_{2N} we apply exactly the same argument as that employed in Helmers, Janssen, and Veraverbeke (1992), at a similar point, to establish that $I_{2N} = O(N^{-1/2})$, a.s. [P], as $N \to \infty$. To show, finally, that

$$I_{1N} = O(N^{-1/2}(\log N)^{3/4})$$

a.s. [P], as $N \to \infty$ we combine the fact that

$$\sup_{|y_{ij}| \leqslant K(\log N)^{1/2}} |\sqrt{N}\,(p_{ij} - H_{F,i,j}(H_{F,i,j}^{-1}(p_{ij}) + y_{ij} N^{-1/2})) + y_{ij} h_{F,i,j}(\xi_{ij}(p_{ij}))|$$
$$= O(N^{-1/4}(\log N)^{3/4})$$

a.s. [P], as $N \to \infty$, for $i = 1, ..., k, j = 1, ..., q$, together with Theorem 1.16 of Götze (1987) with (3.21) and (3.22). ∎

Theorem 3.1 extends results of Singh (1981) and Helmers, Janssen, and Veraverbeke (1992) to a wide class of multivariate generalized quantiles, i.e., to multivariate U-quantiles.

4. Extensions and Examples

It is well known (see, e.g., Bickel and Freedman, 1981) that "the bootstrap commutes with smooth functions." In view of this the following useful result is not very suprising:

Corollary 4.1. *Suppose that $g: \mathbb{R}^{kq} \to \mathbb{R}^k$ is continuously differentiable in a neighborhood of $\xi = (\xi_{ij}(p_{ij}), i = 1, ..., k, j = 1, ..., q)$ with $\xi_{ij}(p_{ij})$ as in* (1.3). *In addition, suppose that the derivative of g at the point ξ is non-zero and that the assumptions of Theorem* 3.1a *are satisfied. Then, for almost every sample sequence $X_1, X_2, ...$*

$$\sup_{x} |P_N^*(\sqrt{N}\,(g(\hat{\xi}^*) - g(\hat{\xi})) \leqslant y) - P(\sqrt{N}(g(\hat{\xi}) - g(\xi)) \leqslant y)|$$
$$\to 0, \quad \text{as} \quad N \to \infty \quad (4.1)$$

Proof. Following the pattern of the proof given by Bickel and Freedman (1981), (4.1) is easily verified using (3.2), (2.3), a Taylor expansion argument, and the fact that $\hat{\xi} \to \xi$, a.s. [P], as $N \to \infty$ (cf. Corollary 3.2 of Helmers, Janssen, and Serfling (1988)) as well as

$$\hat{\xi}^* - \hat{\xi} \to 0 \quad \text{in} \quad P^*\text{-probability} \tag{4.2}$$

a.s. [P], as $N \to \infty$. To verify (4.2) we need the requirement that ξ be uniquely determined. But this is an easy consequence of the local smoothness assumption on H_F. ▮

The a.s. rate at which the sup in (4.1) approaches zero is easily checked to be $O(N^{-1/4}(\ln N)^{3/4})$, under somewhat more stringent smoothness assumptions on g and H_F. Because all of this is fairly straightforward in view of Theorem 3.1(b) and Corollary 4.1 we omit further details.

A second extension is obtained by allowing the kernels h_j, $j = 1, ..., q$, to have possibly different degrees $m_1, ..., m_q$. More generally, we may even let the degree m of $h_j(X_{i1}, ..., X_{im})$, $i = 1, ..., k, j = 1, ..., q$ (cf. (1.2)) not only depend on j, but also on i. It is easily verified that all the results of this paper remain valid, without any further changes. Clearly, in this way the range of applications is considerably enhanced.

Next we discuss a few specific examples of multivariate U-quantiles. In the first of these we take $q = 1$, $m = 1$, and $h_1(x) = x$ and obtain the k-vector of ordinary marginal pth sample quantiles. Similarly, by taking $q = 1$, $m = 2$, $h_1(x_1, x_2) = (x_1 + x_2)/2$ we obtain the k-vector of marginal Hodges–Lehmann location estimators, whereas the choice $q = 1$, $m = 2$, $h_1(x_1, x_2) = |x_1 - x_2|$ gives us the k-vector of estimators of spread proposed by Bickel and Lehmann (1979). In each of these cases Theorem 3.1 tells us that the bootstrap works.

A second type of examples is obtained by taking $q = 2$. For example, let us take all p_{ij}'s equal to $\frac{1}{2}$, $m = 2$, $h_1(x_1, x_2) = (x_1 + x_2)/2$, and $h_2(x_1, x_2) = |x_1 - x_2|$. In this setup $\hat{\xi}$ becomes a vector which consists of k pairs of (marginal) estimators; the first component is the Hodges–Lehmann location estimator and the second one the Bickel–Lehmann estimator of spread. Again, Theorem 3.1 can be employed to find that the bootstrap approximation is asymptotically valid in this case too.

Our third example gives an application of Corollary 4.1. We take $q = 3$, $m = 1$, $h_j(x) = x$, $j = 1, 2, 3$, and $p_{i1} = \frac{1}{2}$, $p_{i2} = \frac{1}{4}$, $p_{i3} = \frac{3}{4}$, $i = 1, ..., k$, and let g: $\mathbb{R}^{3k} \to \mathbb{R}^k$ be the map which sends the ith marginal $(\xi_{i1}, \xi_{i2}, \xi_{i3})$ of ξ into $\xi_{i1}/(\xi_{i3} - \xi_{i2})$, $i = 1, ..., k$. The resulting estimator is the k-vector of marginal sample medians divided by the marginal sample interquartile ranges. Corollary 4.1 asserts that the bootstrap also works here. Of course one can easily supplement this example with many others: e.g., one may consider a linear combination of a fixed number of marginal ordinary sample quantiles in each component or—to give another example—

consider the quotient of the Bickel-Lehmann estimator of spread and the Hodges–Lehmann location estimator in each component (i.e., a generalized "coefficient of variation").

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